Automorphic Forms

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Our references are to the relevant Definitions in [GH24]. We restrict to the number field case, F. Let G be a linear algebraic reductive group over F.

A word on motivation Automorphic forms are blessed and cursed with the property of being interesting to many people for many reasons. This makes them intrinsically interesting, why are they so interesting to different people doing seemingly different things? It also makes it very hard to track down the a broad conceptual understanding of why this particular definition came to the fore. Indeed there are many competing definitions all more or less the same, and the history quickly becomes a tangled mess of dozens of (brilliant) mathematicians conversing and studying these different things. The Langlands program as it appears to me is a promise of unifying all of these things into one theory of immense scope and difficulty.

I will mention one concrete motivation that at least means something to me.

Theorem ([GH24] 4.9, 8.3.5, 10.6.1)**.** *All smooth irreducible representations of $G(F)$ is a subquotient of a parabolically induced representation from a Levi. Any irreducible subquotient of such a parabolically induced representation is an automorphic representation and moreover all automorphic representations appear in this way.*

If (π, V) is a (adjective) representation with Jordan Holder series $(V_i)_{i\in I}$ then a subquotient is a (adjective) representaition isomorphic to V_i/V_{i-1} for some i [GH24][7.6.6]. In this way we see that studying irreducible representations of the points of reductive groups is the same as studying automorphic representations of those points.

1 Archimedean Places

We define an automorphic form on the archimedean places of a group first. The example to keep in mind is when $F = \mathbb{Q}$ in which case there is only one infinite place and this gives $F_\infty = \mathbb{R}$.

Let ν be an archimedean place. Then F_ν is either $\mathbb{R}$ or $\mathbb{C}$. In particular $G(F_\nu)$ is a Lie group and we call a function ***smooth*** $\varphi : G(F_\nu) \to \mathbb{C}$ if it is smooth in the sense of manifolds.

To define moderate growth we fix an embedding $\imath : G \to GL_n$ and then the embedding $G \to SL_{2n}$ via

$$g \mapsto \begin{pmatrix} \imath(g) & \\ & (\imath(g))^{-t} \end{pmatrix}$$

We say that such a function $\varphi : G(F_\infty) = G(\prod_{v|\infty} F_\nu) \to \mathbb{C}$ is of ***moderate growth*** if there are constants $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \le c\|g\|^r = c\left(\prod_{v|\infty} \sup_{1\le i,j\le 2n} |\imath(g)_{i,j,\nu}|_\nu\right)^r$$

this is taking the maximum of the $2n \times 2n \times |\infty|$ three dimensional matrix.

We have already discussed the Lie algebra of G, $\mathfrak{g}$, we denote by $Z(\mathfrak{g})$ the center of the universal enveloping algebra of the complexification of $\mathfrak{g}$. A vector in a $Z(\mathfrak{g})$ module $\varphi \in V$ is called $Z(\mathfrak{g})$-***finite*** if the space $Z(\mathfrak{g})\varphi$ is finite dimensional.

Let $K_\infty \le G(F_\infty)$ be a maximal compact subgroup. Now consider an irrep of K_∞, σ. Given a rep of K (π, V) we denote $V(\sigma)$ the sum of all subrepresentations of V isomorphic to σ. Then we consider $\hat{K}$ the collection of irreducible Hilbert space representations. Then an element $\varphi \in V$ is called K_∞***-finite*** if it is in the set

$$V_{fin} \quad := \quad \oplus_{\sigma\in\hat{K}} V(\sigma)$$

there was some controversy over this definition

To define automorphic forms we look at the representation $C^\infty(F_\infty)$ with the left regular action. In particular the $Z(\mathfrak{g})$ module structure is induced from the action of $\mathfrak{g}$ on $C^\infty(G(F_\infty))$ by

$$z.F(g) = \frac{\partial}{\partial t} F(ge^{tz})$$

Definition. *Let* $\Gamma \le G(F_\infty)$ *some arithmetic subgroup, an automorphic form for* Γ *is a smooth function of moderate growth*

$$\varphi : G(F_\infty) \to \mathbb{C}$$

that is K_∞ *and* $Z(\mathfrak{g})$ *finite with a (left)* Γ *invariance.*

2 (Elliptic) Modular Forms as Archimedean Automorphic Forms

One might ask if there is a special case in which these automorphic forms yeild modular forms. In fact no, the space of automorphic forms is larger than just modular forms, however it gives the space of Maas forms (or modular and Maas forms, depending on convention). We follow [Bum97][3.2] [Eme] and [Boo] for the exposition here.

Recall the definition of a modular form

Definition ([DS05] 1.1.2)**.** *A function*

$$\varphi : \mathcal{H} \to \mathbb{C}$$

that is holomorphic, satisfies

$$\varphi(\gamma.z) = (cz + d)^k \varphi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z})$$

and extends holomorphically to ∞ *is called a modular form of weight* k*.*

These are modular forms with trivial character and full level.

Now give a function on a set X and an action of a group G on X, there is a general way of associating to $\mathrm{Hom}(X, Y)$ a family of maps $\mathrm{Hom}(G, Y)$ indexed by X. This is a manifestation of the tensor-hom adjunction. Effectively if $f : X \to Y$ the we get a map for each $x \in X$ defined on $f_x : G \to Y$ given by $g \mapsto f(g.x)$.

So for our purposes we are trying to take some subset of functions $\mathcal{H} \to \mathbb{C}$ and shift their domain to the $\mathbb{Q}_\infty = \mathbb{R}$ points of some reductive group. In particular it would be sufficient to find a reductive group with a well defined action on the upper half plane. Well every reductive group has the trivial action so we can always accomplish this. We need however the other automorphy conditions to be satisfied however.

Theorem.

$$\mathcal{H} \cong \mathrm{Gl}_2^+(\mathbb{R})/A_{\mathrm{Gl}_2} SO_2(\mathbb{R})$$

as topological spaces. Where $A_{\mathrm{Gl}_2} = \{diag(r, r) : r \in \mathbb{R}^+\}$

Proof. First consider the action

$$\mathrm{Sl}_2(\mathbb{R}) \curvearrowright \mathcal{H} : \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

Then look at the orbit of i, namely

$$\begin{pmatrix} a & b \\ & d \end{pmatrix}.i = \frac{ai+b}{d} = a^2 i + ab$$

which letting $a, b \in \mathbb{R}$ vary is clearly surjective onto the whole upper half plane. So there is one orbit, and hence becuase $\mathrm{Sl}_2(\mathbb{R}) \subseteq \mathrm{Gl}_2^+(\mathbb{R})$ there is one orbit under the same action of $\mathrm{Gl}_2^+(\mathbb{R})$. Thus by the orbit stabiliser we know that

$$\mathcal{H} \cong \mathrm{Gl}_2^+(\mathbb{R})/stab(i)$$

so we want to find

$$stab(i) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2^+(\mathbb{R}) : g.i = i \right\}$$

in particular we solve

$$\begin{aligned} i &= g.i \\ &= \frac{ai+b}{ci+d} \\ &= (c^2+d^2)^{-1}(ai+b)(d-ci) \\ &= (c^2+d^2)^{-1}(ac+bd+i\det(g)) \end{aligned}$$

So equating coefficients we have

$$\det g(c^2+d^2)^{-1} = 1 \implies c^2+d^2 = \det g$$

on the other hand

$$ac+bd = 0$$

Now the pairs $c^2+d^2 = \det g$ are parametrised by $\theta \in [0, 2\pi), r \in \mathbb{R}^+$ using $c = r\sin\theta, d = r\cos\theta$ such that $r^2 = \det g$ hence subbing this into the above equation

$$\frac{-b}{a} = \tan\theta$$

and so $b = -k\sin\theta, a = k\cos\theta$ for some $k \in \mathbb{R}$ but the determinant must be r^2 so $k = r$. Hence

$$stab(i) = \left\{ r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi), r \in \mathbb{R}^+ \right\} = A_{\mathrm{Gl}_2} SO_2(\mathbb{R})$$

This was as sets and we have not checked that the maps are continuous, but all maps are continuous bro dont worry.

Something that other sources never mention, but that seems far from obvious is that

Lemma. Gl_2^+ *is a reductive group over* $\mathbb{Q}$ *.*

Proof. It is the connected component of the identity and therefore a closed subgroup.

references for these bold claims?

Therefore by Matsushima's criterion ([Arz05] for references) we have that $\mathrm{Gl}_2 / \mathrm{Gl}_2^+$ is affine iff Gl_2^+ is reductive. But the thing on the left is the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ which is affine.

As a special case of the above constructions we also have that $\mathcal{H} \cong \mathrm{Sl}_2(\mathbb{R})/SO_2(\mathbb{R})$. Again Sl_2 is reductive. This decomposition of the upperhalf plane suggests that function on it might have some invariance along the maximal compact subgroup of the reductive group Sl_2, which smells of our automorphy condition. Now if we were to push our modular forms along this isomorphism it would with the construction that we outlined earlier in terms of a group action on a set. This is merely *evidence* that if we were to change our modular forms to functions on the reductive groups Sl_2 and Gl_2^+ they may preserve *some* of that invariance and indeed be K-finite. In fact modular forms are automorphic forms for both of these groups. We carry out the proof for the smaller of the two.

$$\begin{tikzcd}
\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} SO_2(\mathbb{R}) = \mathrm{Sl}_2(\mathbb{R}) \arrow[r, "proj"] \arrow[dr, "\text{descend???}"'] & \mathrm{Sl}_2(\mathbb{R})/SO_2(\mathbb{R}) \arrow[r, "\sim", "x\mapsto x.i"'] & \mathcal{H} \\
& \mathrm{Sl}_2(\mathbb{Z}) \setminus \mathrm{Sl}_2(\mathbb{R}) &
\end{tikzcd}$$

Is there a cannonical way to descend here? Averaging is what I would think of, I guess the quotient written there is compact? Is this what they have done though? If I average a modular form do I get the formula below back? No there is no dependence on the input g if you average, also there are problems of convergence that are a mess.. It cant just be ad hoc can it...? This factor came from trivialising a line bundle apparently???

Now applying our construction gives something that is not $\mathrm{Sl}_2(\mathbb{Z})$ invariant so we add a prefactor to ensure this in our automorphic form: Let f be a modular form of weight k then we associate the following function on $\mathrm{Sl}_2(\mathbb{R})$

$$F(g) \quad := \quad (ci+d)^{-k} f(g.i)$$

This looks pretty smooth. The $\mathrm{Sl}_2(\mathbb{Z})$ invariance is obvious from the modularity condition. Note that if we were to do this for Gl_2^+ we would multiply by a prefactor of the determinant. It remains to show the three other properties:

Lemma. *$F(g)$ is of moderate growth.*

Proof. Unraveling the definitions we require two constants such that

$$|F(g)| = |ci+d|^{-k}|f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r$$

A direct computation shows that

$$Im(g.i) = |ci+d|^{-2}$$

hence we require to show

$$Im(g.i)^{k/2}|f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r$$

Somehow invoke polynomial growth...?

Lemma. *$SO_2(\mathbb{R})$ is a maximal compact subgroup inside $\mathrm{Sl}_2(\mathbb{R})$. F is an $SO_2(\mathbb{R})$ finite function.*

Proof. First take $\kappa = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K = SO_2(\mathbb{R})$ then

$$\kappa.i = \frac{i\cos\theta - \sin\theta}{i\sin\theta + \cos\theta} = \frac{-i(-\cos\theta - i\sin\theta)}{e^{i\theta}} = i$$

hence for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{R})$ we have that

$$\begin{aligned} F(g\kappa) &= ((c\cos\theta + d\sin\theta)i - c\sin\theta + d\cos\theta)^{-k} f(g.\kappa.i) \\ &= ((c\cos\theta + d\sin\theta)i - c\sin\theta + d\cos\theta)^{-k} f(gi) \\ &= (ci\cos\theta - c\sin\theta + d\cos\theta + di\sin\theta)^{-k} f(gi) \\ &= (-i^2(ci\cos\theta - c\sin\theta) + de^{i\theta})^{-k} f(gi) \\ &= (ice^{i\theta} + de^{i\theta})^{-k} f(gi) \\ &= (ic + d)^{-k} e^{-ik\theta} f(gi) \\ &= e^{-ik\theta} F(g) \end{aligned}$$

Now this shows that $F(g)$ is acted on by K via a one dimensional and hence irreducible representation. So if $M_k(\mathrm{Sl}_2(\mathbb{R})) \subseteq L^2(\mathrm{Sl}_2(\mathbb{R}))$ is the subrepresentation (scalar multiples of modular forms are modular forms) of functions induced from modular forms then we have shown that $M_k(\mathrm{Sl}_2(\mathbb{R}))$ decomposes as a direct sum over the irreducible representation $\rho : K \to \mathbb{C}^*, \theta \mapsto e^{i\theta}$. Thus

$$F \in L^2(\mathrm{Sl}_2(\mathbb{R}))(\rho)$$

and is therefore K finite.

Lemma. *F is a $Z(\mathfrak{sl}_2)$ finite function.*

Proof. Only a sketch.

The center of the universal enveloping algebra of the complexified Lie algebra is generated by the Casimir operators. From [Gar10] we know that the casimir is

$$\Omega = \frac{1}{2}H^2 + XY + YX$$

we have the coordinates on $\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} SO_2(\mathbb{R}) = \mathrm{Sl}_2(\mathbb{R})$ from [Bum97][1.19 pg 139] in which the casimir acts as the differential operator

$$\Delta = y^2 \left(\left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 \right) - y \frac{\partial^2}{\partial x \partial \theta}$$

[Bum97][1.29 pg 143 ,Prop 2.2.5 pg 155]. Now we claim that F is an eigenfunction for this operator. An element $(x, y, \theta) := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \kappa_\theta \in \mathrm{Sl}_2(\mathbb{R})$ acts on i by sending it to $x + iy$ (elementary computation). The bottom row of the product is $y^{-1/2}\sin\theta; y^{-1/2}\cos\theta$ which results in

$$F(x, y, \theta) = y^{k/2} e^{-ik\theta} f(x + iy)$$

It is then a calculus exercise to apply Δ to this, using the holomorphicity we also get that $f_{xx} - f_{yy} = 0$ and $f_y = if_x$ which cancels away terms and we get that

$$\Delta F(x, y, \theta) = \frac{k}{2} \left(\frac{k}{2} - 1 \right) F(x, y, \theta)$$

Therefore the dimension of $Z(\mathfrak{g})F$ is simply one.

3 Adelic

Similar to the archimedean place we have that

Definition (6.5)**.** *If F is a number field then a smooth function*

$$\varphi : G(\mathbb{A}_F) \to \mathbb{C}$$

is an automorphic form on G if it is smooth, $G(F)$ left invariant, K-finite, $Z(\mathfrak{g})$-finite and of moderate growth.

we will now clarify what these words mean in this setting

If k is a non-archimedian local field then $G(k)$ is totally disconnected and we say that

$$f : G(k) \to \mathbb{C}$$

is smooth if it is locally constant in the induced topology on $G(k)$ from the topology on k.

For the non-archimedian places we define

$$C^\infty(\mathbb{A}_F^\infty) \quad := \quad \bigotimes_{\nu\nmid\infty}{}' C^\infty(G(F_\nu))$$

And for the archimedian places we define

$$C^\infty(G(F_\infty)) \quad := \quad C^\infty\left(\prod_{\nu|\infty} G(F_\nu)\right)$$

For the full Adele we define

$$C^\infty(\mathbb{A}_F) \quad := \quad C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty))$$

A function is ***smooth*** if it is in this set. Note that this gives functions with codomain being the tensor product of a bunch of $\mathbb{C}$'s over $\mathbb{C}$ which is isomorphic to $\mathbb{C}$, so we are justified in making this identification.

A function

$$\varphi : G(\mathbb{A}_F) \to \mathbb{C}$$

is (left) ***invariant*** under the action of a subgroup $H \leq G(\mathbb{A}_F)$ when $\forall \gamma \in H$ we have that

$$\varphi(\gamma g) = g \quad \forall g \in G(\mathbb{A}_F)$$

For the above Definitions we view $G(F) \leq G(\mathbb{A}_F)$ via the diagonal map.

The definition of ***moderate growth*** carries over verbatim, however we change the set of places multiplied over to be all of them now. Note that we have made some choices of embeddings here however the class of functions that is of moderate growth is actually independent of the embedding. $Z(\mathfrak{g})$-***finite*** again translates directly, with the representation now being on smooth *adelic* functions.

The condition of K-finite needs some comment. We choose two subgroups this time; $K_\infty \leq G(F_\infty), K^\infty \leq G(\mathbb{A}_F^\infty)$ where as before K_∞ is a maximal compact subgroup, and K^∞ is some compact open subgroup. We then define $K = K_\infty K^\infty$ the direct product. We then say that a function $f : G(\mathbb{A}_F) \to \mathbb{C}$ is ***K-finite*** if

$$\dim[span_{\mathbb{C}}\{x \mapsto f(xk) : k \in K\}] < \infty$$

This is independent of the choice of K^∞ and independent up to isomorphism of the choice of K_∞.

Remark. In the archimedain subcase [GH24] gives explicitly that the functions are invariant under some arithmentic subgroup. The general Definition of automorphic form does not have this restriction. Moreover the choice of K does not effect the collection of automorphic forms. The correct analogie is that if we required the functinos to be K_∞ invariant functions. Then we recover the more familar notion, in particular modular forms etc.

4 (Elliptic) Modular Forms as Adelic Automorphic Forms

Theorem.

$$\mathrm{Sl}_2(\mathbb{Z}) \setminus \mathrm{Sl}_2(\mathbb{R})/SO_2(\mathbb{R}) \cong Z(\mathbb{A})\,\mathrm{Gl}_2(\mathbb{Q}) \setminus \mathrm{Gl}_2(\mathbb{A})/\,\mathrm{Gl}_2(\hat{\mathbb{Z}})SO_2(\mathbb{R})$$

as topological spaces.

Where $\mathcal{H}$ is the upper half plane, $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $Z(\mathbb{A})$ is the center of $\mathrm{Gl}_2(\mathbb{A})$.

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